

# The asymptotic spectrum of the $\mathcal{N} = 4$ super-Yang-Mills spin chain

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**ABSTRACT:** In this paper we discuss the asymptotic spectrum of the spin chain description of planar  $\mathcal{N} = 4$  SUSY Yang-Mills. The states appearing in the spectrum belong to irreducible representations of the unbroken supersymmetry  $SU(2|2) \times SU(2|2)$  with non-trivial extra central extensions. The elementary magnon corresponds to the bifundamental representation while boundstates of  $Q$  magnons form a certain short representation of dimension  $16Q^2$ . Generalising the Beisert's analysis of the  $Q = 1$  case, we derive the exact dispersion relation for these states by purely group theoretic means.

**KEYWORDS:** Extended Supersymmetry, Integrable Field Theories, AdS-CFT Correspondence, Exact S-Matrix.

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The emergence of integrability on both sides of the AdS/CFT correspondence [1–3] continues to provide improvements in our understanding of large- $N$  gauge theory and string theory. Recent progress has centered on a particular limit [4] where the spin chain describing the single trace operators of  $\mathcal{N} = 4$  SUSY Yang-Mills theory becomes infinitely long.<sup>1</sup> Specifically one considers a limit where the  $U(1)$  R-charge  $J_1$  and scaling dimension  $\Delta$  of the operator become large with the difference  $E = \Delta - J_1$  and the 't Hooft coupling  $\lambda$  held fixed. In this limit, the spectrum corresponds to localised excitations which propagate almost freely on the infinite chain. The remaining interactions between these excitations are governed by a factorisable S-matrix. In this paper we will describe the minimal possibility for the complete spectrum of asymptotic states of the spin chain.<sup>2</sup>

The asymptotic states mentioned above correspond to local excitations above the ferromagnetic groundstate of the spin chain. The latter state corresponds to the gauge theory operator  $\text{Tr}(Z^{J_1})$  where  $Z$  is a complex adjoint scalar field with R-charge  $J_1 = 1$ . The ferromagnetic groundstate is not invariant under the full superconformal algebra  $\mathfrak{psu}(2, 2|4)$ , but instead is only preserved by the subalgebra  $(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)) \ltimes \mathbb{R}$ . The residual symmetry algebra can also be understood as two copies of  $\mathfrak{su}(2|2)$  with their central charges identified. This common central charge will play the role of Hamiltonian for the associated spin chain whose eigenvalue is identified with the combination  $\Delta - J_1$ . Moreover as noted in [6], an important subtlety arising is that this symmetry algebra needs to be further extended by two additional central charges in order to describe excitations of non-zero momenta. This extended unbroken symmetry is linearly realised on excitations above the groundstate which consequently form representations of the corresponding non-abelian symmetry group  $(PSU(2|2) \times PSU(2|2)) \ltimes \mathbb{R}^3$ . In the following we will determine which representations appear in the spectrum of asymptotic states.

The fundamental excitation of the spin chain, known as the magnon, corresponds to an insertion of a single impurity, with definite momentum  $p$ , into the groundstate operator  $\text{Tr}(Z^{J_1})$ . There are a total of sixteen possible choices for the impurity corresponding to the various scalars and spinor fields and covariant derivatives of the  $\mathcal{N} = 4$  theory [7]. As we review below, these excitations fill out a multiplet in the bifundamental representation of  $(PSU(2|2) \times PSU(2|2)) \ltimes \mathbb{R}^3$ . In terms of the centrally-extended algebra described above,

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<sup>1</sup>The importance of this limit was also stressed earlier in [5].

<sup>2</sup>The issue of completeness will be discussed further below.

these are short representations with an exact BPS dispersion relation which is uniquely given by the closure of the algebra to be [6, 8, 9],

$$E = \Delta - J_1 = \sqrt{1 + 8g^2 \sin^2 \left(\frac{p}{2}\right)}. \tag{1}$$

Here, following the convention of [6], we have introduced a coupling  $g$  which is related to the 't Hooft coupling  $\lambda$  by  $g^2 = \lambda/8\pi^2$ . As the residual symmetry generators commute with the Hamiltonian of the spin chain, each state in the multiplet has the same dispersion relation (1). With this in mind, we can think of the sixteen states in the bifundamental multiplet as distinct ‘‘polarisations’’ of a single excitation.

The full spin chain for the planar  $\mathcal{N} = 4$  theory has a closed subsector, known as the  $SU(2)$  sector, where only impurities corresponding to one complex adjoint scalar field are included. Equivalently, we restrict our attention to magnons of a single polarisation. Within this subsector, it is known that the asymptotic spectrum also includes an infinite tower of magnon boundstates [10]. These excitations are labelled by a positive integer  $Q$ , which corresponds to the number of constituent magnons of different flavours, as well as their conserved momentum  $p$ . The location of the corresponding poles in the exact magnon S-matrix indicates that these states have an exact dispersion relation of the form,

$$E = \Delta - J_1 = \sqrt{Q^2 + 8g^2 \sin^2 \left(\frac{p}{2}\right)} \tag{2}$$

which generalises (1). The corresponding classical string solution which precisely reproduces (2) has been found in [11, 12]. Scattering matrices for these states have recently also been constructed in [13, 14]. In the context of the full model, these asymptotic states in the  $SU(2)$  sector should be particular representatives from complete representations of the symmetry group  $(PSU(2|2) \times PSU(2|2)) \ltimes \mathbb{R}^3$ . In fact, we will see below that the  $Q$ -magnon boundstate lies in a short irreducible representation of dimension  $16Q^2$  [15, 16]. The representation in question can be thought of as a supersymmetric extension of the rank- $Q$  traceless symmetric tensor representation of the unbroken  $SO(4) \simeq SU(2) \times SU(2)$  R-symmetry which is a subgroup of  $(PSU(2|2) \times PSU(2|2)) \ltimes \mathbb{R}^3$ . This particular representation includes the known BPS boundstates of magnons in the  $SU(2)$  sector. An important consistency check is that the representation does not lead to boundstates in any of the other rank one subsectors which are known to be absent [4]. In [6], the dispersion relation (1) for excitations transforming in the bifundamental representation of  $(PSU(2|2) \times PSU(2|2)) \ltimes \mathbb{R}^3$  was derived from purely group theoretical means. As an additional test of our results, we will extend the analysis to the symmetric tensor representations relevant for the boundstates described above to provide a parallel group theoretic derivation of the dispersion relation (2).

To begin, let us first focus on a single copy of  $\mathfrak{su}(2|2) \subset (\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)) \ltimes \mathbb{R}$  and review some associated basic facts following [6]. The algebra consists of two bosonic generators  $\mathfrak{L}^\alpha_\beta$  and  $\mathfrak{R}^a_b$  which generate  $\mathfrak{su}(2) \times \mathfrak{su}(2)$  rotations; two fermionic supersymmetry generators  $\mathfrak{Q}^\alpha_b$  and  $\mathfrak{S}^a_\beta$ , and finally the algebra also contains a central charge  $\mathfrak{C}$  which is shared with the other  $\mathfrak{su}(2|2)$ . These generators obey the following (anti-)commutation

relations:

$$[\mathfrak{K}^a_b, \mathfrak{J}^c] = \delta_b^c \mathfrak{J}^a - \frac{1}{2} \delta_b^a \mathfrak{J}^c, \quad (3)$$

$$[\mathfrak{L}^\alpha_\beta, \mathfrak{J}^\gamma] = \delta_\beta^\gamma \mathfrak{J}^\alpha - \frac{1}{2} \delta_\beta^\alpha \mathfrak{J}^\gamma, \quad (4)$$

$$\{\mathfrak{Q}^\alpha_a, \mathfrak{S}^b_\beta\} = \delta_a^b \mathfrak{L}^\alpha_\beta + \delta_\beta^\alpha \mathfrak{K}^b_a + \delta_a^b \delta_\beta^\alpha \mathfrak{C}, \quad (5)$$

where  $\mathfrak{J}$  stands for any generator with appropriate indices.

In addition, as discussed in [6], the  $\mathfrak{su}(2|2)$  algebra is too restrictive for the discussion of excitations with non-zero momentum and it is necessary to enlarge it to  $\mathfrak{su}(2|2) \times \mathbb{R}^2 \cong \mathfrak{psu}(2|2) \times \mathbb{R}^3$ , with two extra central charges  $\mathfrak{P}$  and  $\mathfrak{K}$  satisfying the anti-commutation relations,

$$\{\mathfrak{Q}^\alpha_a, \mathfrak{Q}^\beta_b\} = \epsilon^{\alpha\beta} \epsilon_{ab} \mathfrak{P}, \quad \{\mathfrak{Q}^{\dot{\alpha}}_{\dot{a}}, \mathfrak{Q}^{\dot{\beta}}_{\dot{b}}\} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon_{\dot{a}\dot{b}} \mathfrak{P}, \quad (6)$$

$$\{\mathfrak{S}^a_\alpha, \mathfrak{S}^b_\beta\} = \epsilon^{ab} \epsilon_{\alpha\beta} \mathfrak{K}, \quad \{\mathfrak{S}^{\dot{a}}_{\dot{\alpha}}, \mathfrak{S}^{\dot{b}}_{\dot{\beta}}\} = \epsilon^{\dot{a}\dot{b}} \epsilon_{\dot{\alpha}\dot{\beta}} \mathfrak{K}. \quad (7)$$

The two extra central charges  $\mathfrak{P}$  and  $\mathfrak{K}$  are unphysical in the sense that they vanish when the constraint of vanishing total momentum is imposed.<sup>3</sup> The full extended subalgebra is then obtained by taking direct product between two copies of  $\mathfrak{psu}(2|2) \times \mathbb{R}^3$  and identifying their central charges (both physical and unphysical ones), which extend the residual symmetry algebra from  $(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)) \times \mathbb{R}$  to  $(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)) \times \mathbb{R}^3$ . Under the extended residual symmetry algebra, the central charge  $\mathfrak{C}$  can be identified with the Hamiltonian for the spin chain, whereas the two extra central charges play the role of gauge transformation generators which insert or remove a background chiral field  $Z$  [6].

The fundamental representation of  $\mathfrak{psu}(2|2) \times \mathbb{R}^3 \simeq \mathfrak{su}(2|2) \times \mathbb{R}^2$  corresponds to a 2|2 dimensional superspace given by the basis

$$\square \equiv \begin{pmatrix} \phi^a \\ \psi^\alpha \end{pmatrix}, \quad a = 1, 2, \quad \alpha = 1, 2. \quad (8)$$

Here we have adopted the notation for super Young tableau introduced in [17]. The fields  $\phi^a$  and  $\psi^\alpha$  are bosonic and fermionic, respectively. The group generators acting on this space can be written in the following  $4 \times 4$  supermatrix form:

$$\begin{pmatrix} \mathfrak{K}^a_b & \mathfrak{Q}^\alpha_b \\ \mathfrak{S}^a_\beta & \mathfrak{L}^\alpha_\beta \end{pmatrix} \quad (9)$$

where  $\mathfrak{K}$  and  $\mathfrak{L}$  are  $2 \times 2$  hermitian  $SU(2)$  generators, whereas the entries in  $\mathfrak{Q}$  and  $\mathfrak{S}$  are complex Grassmann variables. We can decompose the fundamental representation  $\square$  under the maximal bosonic subgroup  $SU(2) \times SU(2)$  as,

$$\square = \overbrace{(\square, \mathbf{1})}^{\phi^a} \oplus \overbrace{(\mathbf{1}, \square)}^{\psi^\alpha}. \quad (10)$$

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<sup>3</sup>These two extra central charges  $\mathfrak{P}$  and  $\mathfrak{K}$  in fact combine with  $\mathfrak{C}$  to give a vector under group  $SO(1, 2)$  [6, 4].

The canonical action of the  $\mathfrak{psu}(2|2) \times \mathbb{R}^3$  generators on the components  $\phi^a$  and  $\psi^\alpha$  are then given by [6]

$$\mathfrak{Q}^a_\alpha |\phi^b\rangle = \mathbf{a} \delta^b_a |\psi^\alpha\rangle, \tag{11}$$

$$\mathfrak{Q}^a_\alpha |\psi^\beta\rangle = \mathbf{b} \epsilon^{\alpha\beta} \epsilon_{ab} |\phi^b Z^+\rangle, \tag{12}$$

$$\mathfrak{S}^a_\alpha |\phi^b\rangle = \mathbf{c} \epsilon^{ab} \epsilon_{\alpha\beta} |\psi^\beta Z^-\rangle, \tag{13}$$

$$\mathfrak{S}^a_\alpha |\psi^\beta\rangle = \mathbf{d} \delta^\beta_\alpha |\phi^a\rangle, \tag{14}$$

whereas the  $SU(2)$  generators  $\mathfrak{R}$  and  $\mathfrak{L}$  act on bosonic and fermionic components as

$$\mathfrak{R}^a_b |\phi^c\rangle = \delta^c_b |\phi^a\rangle - \frac{1}{2} \delta^a_b |\phi^c\rangle, \tag{15}$$

$$\mathfrak{L}^\alpha_\beta |\psi^\gamma\rangle = \delta^\gamma_\beta |\psi^\alpha\rangle - \frac{1}{2} \delta^\alpha_\beta |\psi^\gamma\rangle. \tag{16}$$

Here  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  and  $\mathbf{d}$  can be expressed as functions of the magnon spectral parameters  $x^+$  and  $x^-$ , which in turns are related to individual magnon momentum  $p$  by

$$\exp(ip) = \frac{x^+}{x^-}. \tag{17}$$

The symbols  $Z^\pm$  in (12) and (13) denote an inserting (+) or a removing (−) of a background  $Z$  field on the right of the excitation  $\phi^a$  or  $\psi^\alpha$ , respectively. It is important to note that the fundamental representation  $\square$  is in fact a short or atypical representation of  $\mathfrak{psu}(2|2) \times \mathbb{R}^3$ , and it satisfies the shortening condition which for this case is given in terms of the three central charges  $\mathcal{C}$ ,  $\mathcal{P}$  and  $\mathcal{K}$  (eigenvalues of  $\mathfrak{C}$ ,  $\mathfrak{P}$  and  $\mathfrak{K}$ , respectively) as [16, 18]

$$\mathcal{C}^2 - \mathcal{P}\mathcal{K} = \frac{1}{4}. \tag{18}$$

Using the explicit parameterisations for the central charges in terms of spectral parameters given in [6], the shortening condition is equivalent to the constraint on the magnon spectral parameters:

$$x^+ + \frac{g^2}{2x^+} - x^- - \frac{g^2}{2x^-} = i. \tag{19}$$

The exact magnon dispersion relation (1) then arises from the protected central charge  $\mathcal{C}$  carried by the fundamental representation  $\square$ .

Let us recall here that, in terms of  $\mathcal{N} = 4$  SUSY Yang-Mills, the elementary excitation of the spin chain corresponds to the insertion of an impurity field with<sup>4</sup>  $\Delta_0 - J_1 = 1$  into  $\text{Tr}(Z^{J_1})$ . In the limit  $J_1 \rightarrow \infty$ , this corresponds to a single magnon propagating over the ferromagnetic groundstate of the infinite chain. There are eight bosonic and eight fermionic impurities which correspond to sixteen different possible polarisations of the magnon. Explicitly, they correspond to different elements of the set  $\{\Phi_i, D_\mu, \Psi_{\alpha\beta}, \Psi_{\dot{\alpha}\dot{\beta}}\}$ . Here  $i, \mu = 1, \dots, 4$  are indices in the vector representation of the two  $SO(4)$  factors left unbroken by the ferromagnetic groundstate. The former is the unbroken R-symmetry of the  $\mathcal{N} = 4$  theory while the latter corresponds to conformal spin. In view of their interpretation as rotations in the dual string geometry, we denote these  $SO(4)_{S^5}$  and  $SO(4)_{\text{AdS}_5}$ ,

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<sup>4</sup>Here  $\Delta_0$  denotes the bare dimension of the inserted field.

Fields	$SU(2)_{S^5,L} \times SU(2)_{AdS_5,R} \times SU(2)_{S^5,R} \times SU(2)_{AdS_5,L}$	$\Delta_0 - J_1$	$\Delta_0 + J_1$
$Z$	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1})$	0	2
$\bar{Z}$	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1})$	2	0
$\Phi_i$	$(\square, \mathbf{1}; \square, \mathbf{1})$	1	1
$D_\mu$	$(\mathbf{1}, \square; \mathbf{1}, \square)$	1	1
$\Psi_{\alpha\beta}$	$(\square, \mathbf{1}; \mathbf{1}, \square)$	1	2
$\Psi_{\dot{\alpha}\dot{\beta}}$	$(\mathbf{1}, \square; \square, \mathbf{1})$	1	2

**Table 1:**  $SU(2)^4$  representations of  $\mathcal{N} = 4$  fields.

respectively. The scalars  $\Phi_i$  and covariant derivatives  $D_\mu$  form a vector representation of each group. We also use the standard isomorphism  $SO(4) \simeq SU(2)_L \times SU(2)_R$  to introduce dotted and undotted spinor indices for each factor. The fermionic fields of the  $\mathcal{N} = 4$  theory, denoted  $\Psi_{\alpha\beta}, \Psi_{\dot{\alpha}\dot{\beta}}$  ( $\alpha, \dot{\alpha} = 1, 2$ ) transform in the appropriate bispinor representations. The quantum numbers of the  $\mathcal{N} = 4$  fields under the bosonic symmetries are summarised in the following table (for more details, see for example [19]).

To interpret the impurities described above in terms of the supergroup  $(PSU(2|2) \times PSU(2|2)) \ltimes \mathbb{R}^3$ , we note that the bifundamental representation is given by the direct product between two copies of fundamental  $\square$  described above,

$$(\square; \square) = (\square, \mathbf{1}; \square, \mathbf{1}) \oplus (\square, \mathbf{1}; \mathbf{1}, \square) \oplus (\mathbf{1}, \square; \square, \mathbf{1}) \oplus (\mathbf{1}, \square; \mathbf{1}, \square). \tag{20}$$

Here we have also decomposed  $(\square; \square)$  in terms of representations of the  $SU(2)^4$  bosonic subgroup of  $(PSU(2|2) \times PSU(2|2)) \ltimes \mathbb{R}^3$ . There are again sixteen components within this decomposition, precisely what one needs to incorporate the elementary excitations listed in table 1. By identifying the four  $SU(2)$  factors in (20), column by column, with the other four in table 1, we can identify each term in (20) with an impurity  $\mathcal{N} = 4$  theory according to,

$$\Phi_i \equiv (\square, \mathbf{1}; \square, \mathbf{1}), \quad D_\mu \equiv (\mathbf{1}, \square; \mathbf{1}, \square), \quad \Psi_{\alpha\beta} \equiv (\square, \mathbf{1}; \mathbf{1}, \square), \quad \Psi_{\dot{\alpha}\dot{\beta}} \equiv (\mathbf{1}, \square; \square, \mathbf{1}). \tag{21}$$

So the sixteen elementary excitations completely fill up the bifundamental representation of  $SU(2|2) \times SU(2|2)$ .

Having treated the case of the elementary magnon, we now proceed to determine the corresponding representations of  $(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)) \ltimes \mathbb{R}^3$  relevant for the magnon boundstates discovered in [10]. The natural starting point for the  $Q$ -magnon boundstate is to consider the tensor product between  $Q$  copies of the elementary magnon representation  $(\square; \square)$  as given in (20). In particular the magnon boundstates should transform in the short irreducible representations under the residual symmetry algebra  $(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)) \ltimes \mathbb{R}^3$ .

As above we will begin by considering a single copy of  $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$  and will start with the simplest case taking the tensor product between two fundamentals  $\square$  as described in (10). In the usual experience of dealing with Lie algebra, one expects that tensoring two or more irreducible representations (e.g., the fundamental representation) would yield direct sum of irreducible representations (including long and short). However, as pointed

out in [16, 18], such multiplet splitting does not happen generally for  $\mathfrak{psu}(2|2) \times \mathbb{R}^3$ . In particular, for the tensor product of two fundamental representations, the splitting into irreducible representations of lower dimensions can only happen if the central charges carried by the two constituent magnons satisfy the “splitting condition”

$$(\mathcal{C}_1 + \mathcal{C}_2)^2 - (\mathcal{P}_1 + \mathcal{P}_2)(\mathcal{K}_1 + \mathcal{K}_2) = 1 \quad \Rightarrow \quad 2\mathcal{C}_1\mathcal{C}_2 - \mathcal{P}_1\mathcal{K}_2 - \mathcal{K}_1\mathcal{P}_2 = \frac{1}{2}. \quad (22)$$

Here  $\mathcal{C}_i$ ,  $\mathcal{P}_i$  and  $\mathcal{K}_i$  are the central charges carried by the constituent magnons  $i = 1, 2$ . Clearly for arbitrary combinations of the central charges, (22) would not be satisfied, hence tensoring two fundamental representations generically gives us a long irreducible representation of sixteen dimensions.

Interestingly, the splitting condition (22) can be satisfied when the spectral parameters obeys the boundstate pole condition established in [10, 13], that is

$$x_1^- = x_2^+. \quad (23)$$

This can be shown by explicitly calculating the expression in (22) using the spectral parameters.

In this special case, the long multiplet of sixteen dimensions splits into direct sum of two short representations of eight dimensions, and we can label them using the branching rules for super Young tableaux worked out in [17],

$$\square \otimes \square = \square\square \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}. \quad (24)$$

The two terms on the r.h.s. of (24) represent distinct irreducible representations of  $\mathfrak{psu}(2|2) \times \mathbb{R}^3$ . The first irreducible representation, denoted  $\square\square$ , corresponds to a symmetrisation of indices for the bosonic components  $\phi^a$ s of each fundamental representation and anti-symmetrisation of indices for the corresponding Grassmann components  $\psi^\alpha$ s. We will call this the “super-symmetric” representation. In contrast, the second term  $\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$  corresponds to a “super-anti-symmetric” representation where the bosonic/fermionic indices are antisymmetrised/symmetrised, respectively. Both of them are in fact short irreducible representations of  $\mathfrak{psu}(2|2) \times \mathbb{R}^3$ , satisfying the shortening condition given in [16] and carrying the protected central charges.

We can further decompose these short representations into representations under its  $SU(2) \times SU(2)$  bosonic subgroup.<sup>5</sup> In terms of standard  $SU(2)$  Young tableaux the decompositions are

$$\square\square = (\square\square, \mathbf{1}) \oplus (\square, \square) \oplus (\mathbf{1}, \mathbf{1}), \quad (25)$$

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} = (\mathbf{1}, \mathbf{1}) \oplus (\square, \square) \oplus (\mathbf{1}, \square\square). \quad (26)$$

The generalisation to the physical case with two factors of  $\mathfrak{psu}(2|2) \times \mathbb{R}^3$  with their central charges identified is straightforward. Combining (20) and (24), the tensor product

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<sup>5</sup>In fact, our situation is further simplified as the subgroups only involve  $SU(2)$ s, whose Young tableaux only contain single rows at most.

of two bifundamental representations can be decomposed as

$$(\square; \square) \otimes (\square; \square) = (\square\square; \square\square) \oplus (\square\square; \square) \oplus (\square; \square\square) \oplus (\square; \square). \quad (27)$$

Each irreducible representation in the decomposition in (27) is manifestly supersymmetric, containing equal number of bosonic and fermionic components. To identify the nature of the corresponding states, it is convenient to further decompose each term in the decomposition (27) into the irreducible representations of the four SU(2) subgroups. For example, the first term yields,

$$\begin{aligned} (\square\square; \square\square) &= (\square\square, \mathbf{1}; \square\square, \mathbf{1}) \oplus (\square\square, \mathbf{1}; \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}; \square\square, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1}) \\ &\quad \oplus (\square, \square; \mathbf{1}, \mathbf{1}) \oplus (\square, \square; \square\square, \mathbf{1}) \\ &\quad \oplus (\mathbf{1}, \mathbf{1}; \square, \square) \oplus (\square\square, \mathbf{1}; \square, \square) \\ &\quad \oplus (\square, \square; \square, \square). \end{aligned} \quad (28)$$

As each state in the constituent bifundamental multiplet corresponds to an insertion of a particular impurity in the  $\mathcal{N} = 4$  theory, we can identify the terms on the r.h.s. of (28) with appropriate bilinears in the  $\mathcal{N} = 4$  fields. In the appendix, we have listed the SU(2)<sup>4</sup> quantum numbers of arising from each product of two  $\mathcal{N} = 4$  impurities. Comparing (28) with the results in the appendix, we identify the relevant bilinears as,

$$(\square\square; \square\square) \equiv (\Phi_i \otimes \Phi_j) \oplus (\Phi_i \otimes \Psi_{\alpha\beta}) \oplus (\Phi_i \otimes \Psi_{\dot{\alpha}\dot{\beta}}) \oplus (D_a \otimes \Phi_i). \quad (29)$$

where appropriate (anti-)symmetrisations over indices is understood.

As explained above, the two magnon boundstates in the SU(2) sector must correspond to (at least) one of the short representations of  $(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)) \ltimes \mathbb{R}^3$  appearing in the decomposition (28). To identify the relevant representation we note that each magnon of the SU(2) sector carries one unit of a second U(1) R-charge denoted  $J_2$  in [10]. The charge  $J_2$  corresponds to one Cartan generator of the unbroken R-symmetry group  $SO(4) \simeq SU(2) \times SU(2) \subset (PSU(2|2) \times PSU(2|2)) \ltimes \mathbb{R}^3$  normalised to that states in the bifundamental representation of SU(2)  $\times$  SU(2) have charges  $-1 \leq J_2 \leq 1$ . The two-magnon boundstate has charge  $J_2 = 2$ . It is straightforward to check that this value is realised in the term  $(\square\square, \mathbf{1}; \square\square, \mathbf{1})$  appearing in the decomposition (28) of the “bi-super-symmetrised” representation  $(\square\square; \square\square)$  of  $(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)) \ltimes \mathbb{R}^3$ . One may also check that the remaining irreducible representations in the decomposition (27) of the tensor product do not contain states with  $J_2 = 2$ .

Summarising the above discussion we deduce that the two magnon boundstate discovered in [10] is one component of a multiplet of states in the  $(\square\square; \square\square)$  of  $(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)) \ltimes \mathbb{R}^3$ . The dimension of this representation is sixty-four, which corresponds to the number of independent polarisations of the two magnon boundstate. The various bilinear impurities corresponding to these polarisations appear in (29). A check on the identification described above is that there are no bilinears involving only either two fermions or two derivatives. This agrees with the known absence of two magnon boundstates in the SU(1|1) and SL(2,  $\mathbb{R}$ ) sectors, respectively [4, 12, 14].



It is straightforward to extend the discussion to the case of general  $Q$ -magnon scattering, now the multiplet splitting condition can be given by

$$\mathcal{C}_Q^2 - \mathcal{P}_Q \mathcal{K}_Q = \frac{Q^2}{4}, \tag{30}$$

where  $\mathcal{C}_Q$ ,  $\mathcal{P}_Q$  and  $\mathcal{K}_Q$  are the central charges carried by the generic long irreducible representation formed by tensor product between  $Q$  fundamentals. This can be satisfied when we impose the boundstate condition

$$x_i^- = x_{i+1}^+, \quad i = 1, 2, \dots, Q - 1. \tag{31}$$

The tensor product between  $Q$  fundamental representations generally consists of direct sum of long representations [16]. In this special limit (31), it can be further decomposed into direct sum of short representations and labelled by the branching rules in [3] as

$$\underbrace{(\square; \square) \otimes \dots \otimes (\square; \square)}_Q = \underbrace{(\square \square \dots \square)}_Q; \underbrace{(\square \square \dots \square)}_Q \oplus \dots, \tag{32}$$

where the dots represents the direct sum of other irreducible representations. In particular, the representation  $\square \square \dots \square$  being again a short representation under  $\mathfrak{psu}(2|2) \times \mathbb{R}^3$  satisfies the shortening condition in [16] and carries protected central charges. Furthermore, by considering the multi-magnon boundstates in the  $SU(2)$  spin chain, we can conclude that the most general  $Q$ -magnon boundstate should be contained in the first term of the decomposition (32), as such term contains a state of highest weight  $Q$ . It should be a straightforward but tedious exercise to decompose  $(\square \square \dots \square; \square \square \dots \square)$  into the irreducible representations of  $SU(2)^4$ , and rewrite the various terms in the decomposition in terms of the  $\mathcal{N} = 4$  SYM fields as we did for the case of  $Q = 2$ . It would also be interesting to identify these different species of boundstates from the poles in their associated scattering matrices [6, 20]. Even though the classification here does not completely rule out the possibility of having boundstates in other irreducible representation at larger  $Q$ , the states in  $(\square \square \dots \square; \square \square \dots \square)$  should be regarded as the minimal set of boundstates in the asymptotic spectrum.

Here we would like to discuss the number of the possible polarisations for a  $Q$ -magnon boundstate. In decomposing the irreducible representations of  $SU(2|2)$  into those of the  $SU(2) \times SU(2)$  subgroup, the valid Young tableau involved should only contain single rows to comply with the usual rules. As the result the decomposition for irreducible representation of our interests terminates after three terms:

$$\underbrace{\square \square \dots \square}_Q = \underbrace{(\square \square \dots \square, \mathbf{1})}_Q + \underbrace{(\square \square \dots \square, \square)}_{Q-1} + \underbrace{(\square \square \dots \square, \mathbf{1})}_{Q-2}. \tag{33}$$

Simple counting shows that there are  $4Q$  states in this decomposition, and for  $(\square \square \dots \square; \square \square \dots \square)$  which contains all possible polarisations for  $Q$ -magnon boundstates, there are  $(4Q)^2 = 16Q^2$  states. This is the degeneracies for a given boundstate charge  $Q$  and it is drastically different from the number of possible out-going states for

$Q$ -magnon scatterings, which goes exponentially with  $Q$ . This concludes our discussion on the representation of the magnon boundstates.

Having worked out the representation, it is rather straightforward to obtain an exact dispersion relation for the general  $Q$ -magnon bound states by extending the arguments in [6]. The idea is that, as we discussed earlier, the energy  $E$  of the magnon boundstate should again be the physical central charge  $\mathcal{C}$  carried by the associated irreducible representation  $(\underbrace{\square \square \dots \square}_Q; \underbrace{\square \square \dots \square}_Q)$  under the extended residual symmetry algebra. Recall that this central charge (along with the two extra ones) is shared between the two  $\mathfrak{su}(2|2)$ s in the extended algebra, in addition, the magnon boundstate transforms under identical short irreducible representation with respect to each  $\mathfrak{su}(2|2)$ . We conclude that it is sufficient to consider the action of only single  $\mathfrak{su}(2|2)$  (with two extra central charges) on the boundstate, and treat the components transforming under the other  $\mathfrak{su}(2|2)$  as the spectators, just like the infinite number of background  $Z$  fields. Moreover, as  $\mathcal{C}$  should commute with other group generators which relate all  $16Q^2$  different polarisations for magnon boundstate of charge  $Q$ , the dispersion relation deduce here would be identical for all of them.

Moving onto a  $Q$ -magnon boundstate which transforms as  $\underbrace{\square \square \dots \square}_Q$  under  $\mathfrak{psu}(2|2) \times \mathbb{R}^3$ , so we have

$$\underbrace{\square \square \dots \square}_Q : |\Xi_Q\rangle \equiv |\xi^{A_1} \xi^{A_2} \dots \xi^{A_{Q-1}} \xi^{A_Q}\rangle, \quad (34)$$

where we have omitted the infinite number of background  $Z$  fields. Here we have also introduced  $\xi^{A_i} = \{\phi^{a_i}; \psi^{\alpha_i}\}$  a generalized vector and  $A_i = \{a_i, \alpha_i\}$  a generalized index for notational conveniences. We are interested in the central charge  $\mathcal{C}_Q$  carried by such state, which would in turn give us the required dispersion relation. This can be obtained by considering the actions from both sides of the commutator (5) on the higher tensor representations, and combining with the algebraic relations (11)–(16).

We shall give our calculational details in the generalized indices  $A_i$  and only focus on the algebraic structures, the explicit conversion into bosonic and fermionic indices,  $a_i$  and  $\alpha_i$  respectively, should be obvious. First let us act the l.h.s. of (5) on  $|\Xi_Q\rangle$  using (11)–(14) to obtain

$$\sum_{i=1}^Q \{\mathfrak{Q}, \mathfrak{S}\}_{A_i}^{B_i} |\Xi_Q^{C_i}\rangle = \sum_{i=1}^Q (\mathbf{a}_i \mathbf{d}_i - \mathbf{b}_i \mathbf{c}_i) \delta_{A_i}^{C_i} |\Xi_Q^{B_i}\rangle + \sum_{i=1}^Q \mathbf{b}_i \mathbf{c}_i \delta_{A_i}^{B_i} |\Xi_Q^{C_i}\rangle. \quad (35)$$

The notation here means that  $\{\mathfrak{Q}, \mathfrak{S}\}_{A_i}^{B_i}$  only acts on the  $i$ -th fundamental representation in the tensor representation and the superscript  $C_i$  in  $|\Xi_Q^{C_i}\rangle$  is also for highlighting such fact.

On the other hand, the action of the r.h.s. of (5) on  $|\Xi_Q\rangle$  gives.

$$\sum_{i=1}^Q (\mathfrak{L} + \mathfrak{R} + \mathfrak{C})_{A_i}^{B_i} |\Xi_Q^{C_i}\rangle = \sum_{i=1}^Q \left\{ \delta_{A_i}^{C_i} |\Xi_Q^{B_i}\rangle + \left( C_i - \frac{1}{2} \right) \delta_{A_i}^{B_i} |\Xi_Q^{C_i}\rangle \right\}, \quad (36)$$

where we have used  $C_i$  to denote the central charge carried by the  $\xi^{A_i}$ , that is  $\mathfrak{C}|\xi^{A_i}\rangle = C_i|\xi^{A_i}\rangle$ . From (35) and (36), we can deduce the closure of the symmetry algebra requires

$$\mathbf{a}_i \mathbf{d}_i - \mathbf{b}_i \mathbf{c}_i = 1 \quad \text{and} \quad C_i = \left( \frac{1}{2} + \mathbf{b}_i \mathbf{c}_i \right), \quad i = 1, \dots, Q, \quad (37)$$

which then implies that  $\mathcal{C}_i = \frac{1}{2}(\mathbf{a}_i \mathbf{d}_i + \mathbf{b}_i \mathbf{c}_i)$ . The central charge  $\mathcal{C}_Q$  of  $|\Xi_Q\rangle$  is given by sum of the individual central charges, so we have

$$\mathcal{C}_Q = \sum_{i=1}^Q \mathcal{C}_i = \frac{1}{2} \sum_{i=1}^Q (\mathbf{a}_i \mathbf{d}_i + \mathbf{b}_i \mathbf{c}_i) = \frac{1}{2} (\mathbf{A} \mathbf{D} + \mathbf{B} \mathbf{C}) . \quad (38)$$

This is the central charge of the  $Q$ -magnon boundstate in terms of  $\mathbf{a}_i$ ,  $\mathbf{b}_i$ ,  $\mathbf{c}_i$  and  $\mathbf{d}_i$ , and here we have also introduced  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  and  $\mathbf{D}$  which should be functions of the spectral parameters for the boundstates  $X^\pm$ . To proceed obtaining the explicit expression for  $\mathcal{C}_Q$ , we need to work out  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  and  $\mathbf{D}$  or at least some combinations of them in terms of the magnon boundstate spectral parameters, this is where the two extra central charges  $\mathfrak{P}$  and  $\mathfrak{K}$  in (6) and (7) come in. First consider the actions of (6) and (7) on  $|\Xi_Q\rangle$ , one can deduce that

$$\mathfrak{P}|\Xi_Q\rangle = \sum_{i=1}^Q \mathbf{a}_i \mathbf{b}_i \prod_{j=i+1}^Q \exp(-ip_j) |\Xi_Q^{C_i} Z^+\rangle , \quad (39)$$

$$\mathfrak{K}|\Xi_Q\rangle = \sum_{i=1}^Q \mathbf{c}_i \mathbf{d}_i \prod_{j=i+1}^Q \exp(ip_j) |\Xi_Q^{C_i} Z^-\rangle . \quad (40)$$

In deducing (39) and (40), we have also used the consistency relation  $|Z^\pm \xi^{A_i}\rangle = \exp(\mp ip_i) |\xi^{A_i} Z^\pm\rangle$  to shift the insertion/removal of  $Z$  field to the far right.

Using the expressions for  $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i$  and  $\mathbf{d}_i$  in [6], we have  $\mathbf{a}_i \mathbf{b}_i = \alpha(\exp(-ip_i) - 1)$  and  $\mathbf{c}_i \mathbf{d}_i = \beta(\exp(ip_i) - 1)$  with  $\alpha$  and  $\beta$  some constants for the time being. The two additional central charges carried by the magnon boundstate are given by

$$\mathcal{P}_Q = \alpha \left( \prod_{i=1}^Q \exp(-ip_i) - 1 \right) = \mathbf{A} \mathbf{B} \quad \text{and} \quad \mathcal{K}_Q = \beta \left( \prod_{i=1}^Q \exp(ip_i) - 1 \right) = \mathbf{C} \mathbf{D} . \quad (41)$$

The momentum carried by the magnon boundstates should be the sum of constituent momenta, this allows us to write down

$$\mathcal{P}_Q = \mathbf{A} \mathbf{B} = \alpha(e^{-iP} - 1) = \alpha \left( \frac{X^-}{X^+} - 1 \right) , \quad \mathcal{K}_Q = \mathbf{C} \mathbf{D} = \beta(e^{iP} - 1) = \beta \left( \frac{X^+}{X^-} - 1 \right) , \quad (42)$$

where  $P = \sum_{i=1}^Q p_i$  is the momentum carried by the  $Q$ -magnon boundstate. When we restrict to the physical states which living in  $\mathfrak{psu}(2|2) \times \mathbb{R}^3$ , both of extra central charges should vanish.

Moreover, as the  $Q$ -magnon boundstates transform in the short representation  $\overline{\mathbb{Z}} \overline{\mathbb{Z}} \cdots \overline{\mathbb{Z}}$  of  $\mathfrak{psu}(2|2) \times \mathbb{R}^3$ , in terms of its central charges  $\mathcal{C}_Q, \mathcal{P}_Q$  and  $\mathcal{K}_Q$ , the shortening condition it obeys is

$$\mathcal{C}_Q^2 - \mathcal{P}_Q \mathcal{K}_Q = \frac{Q^2}{4} . \quad (43)$$

In the light of (18) and (19), this should in turn provide a constraint on the boundstate spectral parameters  $X^\pm$  as

$$X^+ + \frac{g^2}{2X^+} - X^- - \frac{g^2}{2X^-} = iQ . \quad (44)$$

This can be guaranteed and reduced correctly to trivial  $Q = 1$  case if we set

$$\begin{aligned} \mathbf{AD} &= \sum_{i=1}^Q \mathbf{a}_i \mathbf{d}_i = -i(X^+ - X^-), \\ \mathbf{BC} &= \sum_{i=1}^Q \mathbf{b}_i \mathbf{c}_i = i \frac{g^2}{2} \left( \frac{1}{X^+} - \frac{1}{X^-} \right). \end{aligned} \tag{45}$$

Using the explicit expressions for  $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i$  and  $\mathbf{d}_i$  in terms of the magnon spectral parameters given in [6], we deduce that

$$\begin{aligned} X^+ - X^- &= \sum_{i=1}^Q (x_i^+ - x_i^-), \\ \frac{1}{X^+} - \frac{1}{X^-} &= \sum_{i=1}^Q \left( \frac{1}{x_i^+} - \frac{1}{x_i^-} \right). \end{aligned} \tag{46}$$

Combining (46) with (42), they give three constraints on  $\{x_1^\pm, \dots, x_Q^\pm\}$  in terms of  $X^\pm$  which can be satisfied by the combination

$$X^+ = x_1^+, \quad X^- = x_Q^-, \tag{47}$$

$$x_i^- = x_{i+1}^+, \quad i = 1, \dots, Q - 1. \tag{48}$$

The equation (48) is identical to the multiplet splitting condition given earlier (30), as the “super-symmetric” representation  $\boxtimes \boxtimes \cdots \boxtimes$  can only arise from the decomposition of general  $Q$ -magnon tensor product after (30) is imposed.

From (42) and (43) or (44), we can also deduce  $\mathcal{C}_Q$  for the magnon boundstate

$$\mathcal{C}_Q = \frac{1}{2} \sqrt{(\mathbf{AD} - \mathbf{BC})^2 + 4\mathbf{ABCD}} = \frac{1}{2} \sqrt{Q^2 + 16\alpha\beta \sin^2 \left( \frac{P}{2} \right)}. \tag{49}$$

The product  $\alpha\beta$  is in general a function of the 't Hooft coupling  $\lambda$ . For the case of single magnon, it has been set to  $\alpha\beta = \lambda/16\pi^2$  by considering the BMN limit [7], this dependence should interpolate to case of  $Q > 1$ , and indeed one can confirm that for example by considering the Frolov-Tseytlin limit [21] as in [10]. In any case, we deduce that the dispersion relation for the magnon boundstate from the group theoretical means is

$$E = \Delta - J_1 \equiv 2\mathcal{C}_Q = \sqrt{Q^2 + \frac{\lambda}{\pi^2} \sin^2 \left( \frac{P}{2} \right)}. \tag{50}$$

This formula reduces the one proposed in [10] for single magnon boundstate of charge  $Q = 1$ , with  $\Delta - J_1$  coincides with (1). It is also important to note that, as discussed earlier, there will be  $16Q^2$ -fold degeneracies which correspond to the all possible polarisations of a  $Q$ -magnon boundstate, all share the same dispersion relation (1).

We would also like to make a comment on the case when there are more than one boundstate in the asymptotic spin chain, namely a state of the form  $|\Xi_{Q_1} \dots \Xi_{Q_M}\rangle$ . Here

$M$  is the number of the boundstates each of which are well-separated, and  $Q_k$  is the number of constituent magnons in the  $k$ -th boundstate. In this case the dispersion relation (50) is simply generalised to give

$$E = \Delta - J_1 \equiv \sum_{k=1}^M 2\mathcal{C}_{Q_k} \quad \text{with} \quad \mathcal{C}_{Q_k} \equiv \frac{1}{2} \sqrt{Q_k^2 + \frac{\lambda}{\pi^2} \sin^2 \left( \frac{P_k}{2} \right)}, \quad (51)$$

where  $P_k$  is the total momentum of the  $k$ -th boundstate in the asymptotic spin chain.

In this paper we have described the infinite tower of BPS boundstates appearing in the asymptotic spectrum of the  $\mathcal{N} = 4$  spin chain and identified the corresponding representation of supersymmetry in which they transform. As these are short representations we expect that these states are present for all values of the 't Hooft coupling,  $\lambda$ . Indeed, as discussed in [10, 11], the representatives of the boundstate multiplets lying in a given  $SU(2)$  sector are directly visible both in one-loop gauge theory and in semiclassical string theory which correspond to small and large  $\lambda$  respectively. An obvious question is whether additional asymptotic states are also present. At this point we cannot rule out the possibility that some of the additional short representations, which appear in the tensor product of bifundamentals when the shortening condition is obeyed, also correspond to BPS boundstates in the spectrum. However, the representations which can occur are certainly constrained by the known absence of boundstates in the remaining rank one sectors. In particular this rules out additional boundstates with  $Q = 2$ .

In closing, we should note that there are two classes of states which we have not included in our discussion. First, the semiclassical string theory analysis of [4] suggests the presence of an infinite tower of neutral boundstates appearing as poles in the two-magnon S-matrix. These poles should appear at values of the kinematic variables which do not satisfy the shortening condition. In fact, for such generic values of the momenta the tensor product of two bifundamentals actually consists of a single irreducible long multiplet [16]. Each of the neutral boundstates of [4] must therefore fill out such a multiplet. As the energies of these states are not protected, their behaviour away from the region of large 't Hooft coupling is still unclear. Finally we recall that the  $\mathcal{N} = 4$  spin chain also contains a singlet state of zero energy [6]. However, in a crossing invariant theory, this state is indistinguishable from the vacuum.

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## A. Decomposition of $\mathcal{N} = 4$ fields into $SU(2)^4$ representations

Here we list all possible tensor decompositions between two  $\mathcal{N} = 4$  SYM excitations for readers' convenience.

$$\begin{aligned} \Phi_i \otimes \Phi_j & : (\square, \mathbf{1}; \square, \mathbf{1}) \otimes (\square, \mathbf{1}; \square, \mathbf{1}) & (A.1) \\ & = (\square\square, \mathbf{1}; \square\square, \mathbf{1}) \oplus (\square\square, \mathbf{1}; \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}; \square\square, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1}), \end{aligned}$$

$$\begin{aligned} D_a \otimes D_b & : (\mathbf{1}, \square; \mathbf{1}, \square) \otimes (\mathbf{1}, \square; \mathbf{1}, \square) & (A.2) \\ & = (\mathbf{1}, \square\square; \mathbf{1}, \square\square) \oplus (\mathbf{1}, \square\square; \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}; \mathbf{1}, \square\square) \oplus (\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1}), \end{aligned}$$

$$D_a \otimes \Phi_i : (\mathbf{1}, \square; \mathbf{1}, \square) \otimes (\square, \mathbf{1}; \square, \mathbf{1}) = (\square, \square; \square, \square), \quad (A.3)$$

$$\Psi_{\dot{\alpha}\dot{\beta}} \otimes \Psi_{\gamma\delta} : (\mathbf{1}, \square; \square, \mathbf{1}) \otimes (\square, \mathbf{1}; \mathbf{1}, \square) = (\square, \square; \square, \square), \quad (A.4)$$

$$\begin{aligned} \Psi_{\alpha\beta} \otimes \Psi_{\gamma\delta} & : (\square, \mathbf{1}; \mathbf{1}, \square) \otimes (\square, \mathbf{1}; \mathbf{1}, \square) & (A.5) \\ & = (\square\square, \mathbf{1}; \mathbf{1}, \square\square) \oplus (\square\square, \mathbf{1}; \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}; \mathbf{1}, \square\square) \oplus (\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1}), \end{aligned}$$

$$\begin{aligned} \Psi_{\dot{\alpha}\dot{\beta}} \otimes \Psi_{\dot{\gamma}\dot{\delta}} & : (\mathbf{1}, \square; \square, \mathbf{1}) \otimes (\mathbf{1}, \square; \square, \mathbf{1}) & (A.6) \\ & = (\mathbf{1}, \square\square; \square\square, \mathbf{1}) \oplus (\mathbf{1}, \square\square; \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}; \square\square, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1}), \end{aligned}$$

$$\Phi_i \otimes \Psi_{\alpha\beta} : (\square, \mathbf{1}; \square, \mathbf{1}) \otimes (\square, \mathbf{1}; \mathbf{1}, \square) = (\mathbf{1}, \mathbf{1}; \square, \square) \oplus (\square\square, \mathbf{1}; \square, \square), \quad (A.7)$$

$$\Phi_i \otimes \Psi_{\dot{\alpha}\dot{\beta}} : (\square, \mathbf{1}; \square, \mathbf{1}) \otimes (\mathbf{1}, \square; \square, \mathbf{1}) = (\square, \square; \mathbf{1}, \mathbf{1}) \oplus (\square, \square; \square\square, \mathbf{1}), \quad (A.8)$$

$$D_a \otimes \Psi_{\alpha\beta} : (\mathbf{1}, \square; \mathbf{1}, \square) \otimes (\square, \mathbf{1}; \mathbf{1}, \square) = (\square, \square; \mathbf{1}, \mathbf{1}) \oplus (\square, \square; \mathbf{1}, \square\square), \quad (A.9)$$

$$D_a \otimes \Psi_{\dot{\alpha}\dot{\beta}} : (\mathbf{1}, \square; \mathbf{1}, \square) \otimes (\mathbf{1}, \square; \square, \mathbf{1}) = (\mathbf{1}, \mathbf{1}; \square, \square) \oplus (\mathbf{1}, \square\square; \square, \square). \quad (A.10)$$

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